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Baroclinic-type instability in a gas centrifuge heated from above

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A baroclinic-type instability in a gas centrifuge heated from above is discussed. The instability is shown to be of overstable type, and estimates of the growth time scale and oscillation period of the unstable mode are given.

1. Introduction and summary

Let us consider flow in a gas centrifuge heated from above in which the equipotentials of effective gravity are almost vertical, i.e. parallel to the axis of rotation $\mathbf{\Omega}$. As is discussed by Barcilon & Pedlosky (1967), Homsy & Hudson (1969) and Sakurai & Matsuda (1974), there exists a steady state in which the Ekman layers on the top and the bottom pump a weak axial current and the thermal-wind effect causes an axially-differential rotation. In this steady state the isentropic surfaces cross the equipotentials. Therefore we expect an instability different in nature from the predominantly gravitational instability discussed by Homsy & Hudson (1971) for a rotating Boussinesq liquid heated from below, and more like one of the various kinds of baroclinic instability studied in the meteorological literature. It is the purpose of the present paper to justify this expectation.

To understand the basic properties of the expected instability, and to get an idea of how to formulate the idealized mathematical problem, we recapitulate the usual argument showing how baroclinic instability is possible on energetic grounds. Let us consider a virtual displacement by which fluid particle A in figure 1 is exchanged with B. Because A has a smaller density and a lower potential energy, the potential energy of the system is decreased by this displacement. The potential energy lost is available, assuming that the detailed dynamics allow it, for transformation into kinetic energy of an unstable disturbance.

In the usual baroclinic wave instability studied in meteorology (Bretherton 1966; Charney 1947; Eady 1949; Green 1960; Hirota 1968; McIntyre 1970*a*; Pedlosky 1971; Tokioka 1971), the conditions at boundaries not parallel to $\mathbf{\Omega}$ can crucially affect the dynamics because of the importance of vortex stretching. The above energy argument, however, is relevant to other cases as well (Yanai & Tokioka 1969; McIntyre 1970*b*) and suggests possible local instability of other flow configurations in which the insentropes cross the equipotentials, and with dynamics not necessarily rotation dominated. Thus it seems reasonable to look

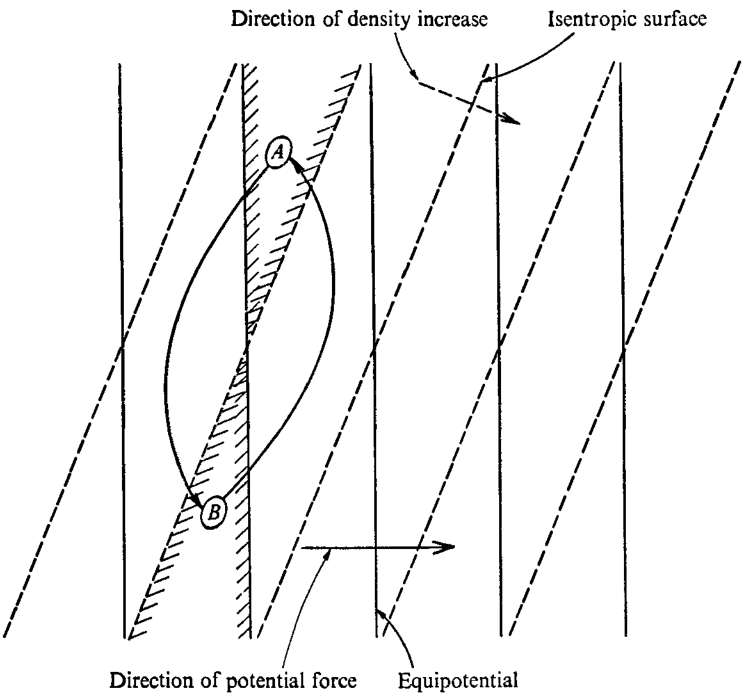

FIGURE 1. Schematic representation of a virtual displacement which is excited at the cost of potential energy. Fluid particles A and B, in the hatched regions, are exchanged in this virtual displacement, and the potential energy of the system is transformed into kinetic energy of the motion.

for a baroclinic-type instability in a gas centrifuge, neglecting in the first instance the (widely separated) top and bottom boundaries.

Taking into account the above-mentioned property of the excitation mechanism, we idealize the problem as follows. Consider a steady state of a rotating gas in which a slight temperature gradient is imposed along the axis of rotation. We take into account the slight differential rotation caused by the thermal-wind effect. We neglect the weak axial flow pumped by the top and the bottom Ekman layers, because it is of order $E^{\frac{1}{2}}$ relative to the thermal-wind velocity, where E is the Ekman number and is small in practical cases ($E = \nu/\Omega H^2$, where ν is the kinematic viscosity, Ω the angular velocity and H a typical dimension of the centrifuge). Our problem is to study the behaviour of an infinitesimal disturbance superposed on the above steady state. We assume that the axial component of the wavenumber vector is large in comparison with H^{-1}. Therefore we neglect the presence of the top and bottom, and moreover approximate the basic state, for some purposes, by values on a certain equatorial plane perpendicular to the axis of rotation. To examine the effect of the side walls, we assume that the flow field is confined in an annular region with inner and outer radii $\tilde{r}_1$ and $\tilde{r}_2$, respectively. Because we neglect viscosity, the boundary condition on the side walls is that the radial component of the velocity vanishes. For the sake of simplicity, and to emphasize the effect of compressibility, the radial density scale height is assumed to be small in comparison with the inner radius $\tilde{r}_1$ and thickness $\tilde{r}_2 - \tilde{r}_1$ of the annulus. Finally, gravity and thermal conductivity are neglected.

As will be clarified in the next section, our expectation that a baroclinic-type instability appears is justified by the solution of our idealized problem. From equation (35) and table 1 in §2, we find that the instability is of overstable type (see, for example, Spiegel 1972), and that the growth time scale and the oscillation period of the unstable mode are of the order of a second and a minute, respectively, for the standard centrifuge (i.e. one with height 1 m, radius 10 cm, mean temperature 20 °C, a temperature difference between the top and bottom of 10 °C, angular velocity 40000 r.p.m. and uranium hexafluoride as the working fluid). We do not have any available experimental data on the flow in a gas centrifuge. This may show that such data are classified as secret or that no reliable data have yet been obtained. Anyway, we believe that the fluctuations with a time scale of minutes observed in such a centrifuge may be identifiable with the wavy disturbances excited by the baroclinic-type instability. These wavy disturbances could take the form of small amplitude laminar motions and not handicap the efficiency of the centrifuge, provided that a certain threshold condition is satisfied. If this condition is violated, however, nonlinear interaction among waves becomes predominant, and eventually turbulent motion will result. In such circumstances, the efficiency of the centrifuge will be reduced appreciably. The determination of this threshold condition and the examination of the details of the nonlinear motions is one of the important problems of the gas centrifuge, but is beyond the scope of the present paper.

2. Analysis of the instability

Let us introduce the following non-dimensional quantities:

$$\left.\begin{aligned} t = \tilde{t}/t_0, \quad r = \tilde{r}/l_0, \quad z = \tilde{z}/l_0, \quad (u,v,w) = v_0^{-1}(\tilde{u},\tilde{v},\tilde{w}), \\ T = \tilde{T}/\tilde{T}_0, \quad p = \tilde{p}/\tilde{p}_s, \quad \rho = \tilde{\rho}/\tilde{\rho}_s, \end{aligned}\right\} \tag{1}$$

where

$$t_0 = \Omega^{-1}, \quad v_0 = (R\tilde{T}_0)^{\frac{1}{2}}, \quad l_0 = v_0 t_0, \tag{2}$$

t is the time, (r,θ,z) a rotating system of cylindrical co-ordinates whose angular velocity is $\mathbf{\Omega} = (0,0,\Omega)$, (u,v,w) the velocity components in this system of co-ordinates, T the temperature, p the pressure, ρ the density and R the gas constant. The suffixes 0 indicate conditions at a typical point ($r = z = 0$) in the flow field and the suffixes s denote a static reference state of rigid-body rotation (not the basic state for the instability). The tildes denote physical, dimensional quantities; $\tilde{T}_0$ is the temperature, taken as constant, of the reference state. Therefore $\tilde{p}_s$ and $\tilde{\rho}_s$ are expressed as follows:

$$\tilde{p}_s = \tilde{p}_0\rho_s, \quad \tilde{\rho}_s = \tilde{\rho}_0\rho_s, \quad \rho_s = \exp(\tfrac{1}{2}r^2), \quad \tilde{p}_0 = \tilde{\rho}_0 R\tilde{T}_0. \tag{3}$$

Assuming that a slight temperature gradient is imposed along the axis of rotation and an accompanying thermal wind exists in the basic state, we express this state, denoted by non-dimensional quantities with suffix B, as

$$\left.\begin{aligned} p = p_B = 1, \quad \rho = \rho_B = 1-\delta z, \quad T = T_B = 1+\delta z, \\ u = u_B = 0, \quad v = v_B = \tfrac{1}{2}\delta rz, \quad w = w_B = 0, \end{aligned}\right\} \tag{4}$$

where

$$\delta = \{\tilde{T}(\tilde{z} = l_0) - \tilde{T}(\tilde{z} = 0)\}/\tilde{T}(\tilde{z} = 0) \ll 1, \tag{5}$$

and terms $O(\delta^2)$† have been neglected. Because the existence of the axial temperature gradient and the thermal wind $\partial v_B/\partial z$, i.e. the non-vanishing of δ, is crucial for the excitation of the baroclinic-type instability, the terms proportional to δ are retained in (4). In virtue of our neglect of the top and bottom boundaries [see also remarks below (12)] we assume that the time-dependent disturbance superposed on the basic steady state is of the form

$$q = \hat{q}(r)\,\epsilon, \quad \epsilon = \epsilon_1 \exp(im\theta + inz - \omega t), \tag{6}$$

where q refers to an arbitrary physical quantity, ϵ_1 is an amplitude parameter which we shall subsequently require to be infinitesimal with respect to δ, n is of order unity and m is an integer of order unity.

Substituting the above expressions into the inviscid perfect-gas equations and picking out terms proportional to ϵ, i.e. linearizing, we obtain

$$0 = -\omega\hat{\rho} + \frac{1}{r\rho_s}\frac{\partial(r\rho_s\hat{u})}{\partial r} + \frac{im\hat{v}}{r} + in\hat{w} \quad \text{(continuity)}, \tag{7}$$

$$\left.\begin{aligned} 0 &= -\omega\hat{u} - 2\hat{v} + r\hat{T} + \partial\hat{p}/\partial r, &(8)\\ 0 &= -\omega\hat{v} + 2\hat{u} + im\hat{p}/r + \tfrac{1}{2}\delta r\hat{w}, &(9)\\ 0 &= -\omega\hat{w} + in\hat{p}, &(10)\end{aligned}\right\} \quad \text{(momentum)},$$

$$0 = -\omega\{\hat{T} - (\gamma-1)\hat{\rho}\} - (\gamma-1)\,r\hat{u} + \delta\gamma\hat{w} \quad \text{(adiabatic)}, \tag{11}$$

$$\hat{p} = \hat{\rho} + \hat{T} \quad \text{(state)}. \tag{12}$$

In the above, the terms proportional to z have been neglected in comparison with those retained, an approximation valid in the neighbourhood of the equatorial plane $z = 0$. For example, $(\delta z/r\rho_s)\,\partial(\rho_s\hat{u}r)/\partial r$ has been neglected in comparison with $(r\rho_s)^{-1}\,\partial(\rho_s\hat{u}r)/\partial r$ in (7). Another example is the neglect of $\delta z\hat{u}$ in comparison with $2\hat{u}$ in (9). This approximation is consistent with our intention of looking for a local kind of baroclinic-type instability.

Solving (9)–(12) for $\hat{u}$, $\hat{v}$, $\hat{w}$ and $\hat{T}$ gives

$$\hat{u} = \frac{\gamma\omega}{(\gamma-1)\,r}\hat{\rho} + \frac{\hat{p}}{(\gamma-1)\,r}\left\{-\omega + \frac{in\gamma\delta}{\omega}\right\}, \tag{13}$$

$$\hat{v} = \frac{2\gamma\hat{\rho}}{(\gamma-1)\,r} + \left\{-\frac{2}{(\gamma-1)\,r} + \frac{im}{r\omega} + \frac{in\delta}{\omega^2}\left(\frac{2\gamma}{(\gamma-1)\,r} + \frac{r}{2}\right)\right\}\hat{p}, \tag{14}$$

$$\hat{w} = in\hat{p}/\omega, \quad \hat{T} = -\hat{\rho} + \hat{p}. \tag{15), (16}$$

Substitution of (13)–(16) into (8) gives

$$\hat{\rho} = \frac{r(d\hat{p}/dr)}{r^2+\gamma_1} + \left\{-\frac{in\delta}{\omega^2} + 1 - \frac{2im + \omega(4+\omega^2)}{\omega(r^2+\gamma_1)}\right\}\hat{p}, \quad \gamma_1 = \frac{\gamma(4+\omega^2)}{\gamma-1}. \tag{17}$$

† Those remembering the notation 'O' imperfectly are recommended to read it as 'a term of order at most . . .'. This will remind them that $f = O(g)$ means that $|f| < A|g|$ for some positive A as the limit is approached (Lighthill 1960).

Substitution of (17) into (13) gives

$$\hat{u} = \frac{\gamma\omega}{(\gamma-1)r}\left[\frac{r}{r^2+\gamma_1}\frac{d\hat{p}}{dr} - \left\{\frac{2im+\omega(4+\omega^2)}{\omega(r^2+\gamma_1)} - \frac{\gamma-1}{\gamma}\right\}\hat{p}\right]. \tag{18}$$

Substitution of (13)–(17) into (7) gives us the following single equation:

$$0 = 4\frac{d^2q}{d\xi^2} + 4A_1\frac{dq}{d\xi} - \left\{\frac{1}{4} + \frac{\omega_1\alpha^2 A_0}{(1+\omega_1)^2} + A_1\right\}q, \tag{19}$$

where

$$A_1 = \gamma_1/\xi(\xi+\gamma_1), \tag{20}$$

$$\begin{aligned} A_0 = {} & \frac{1}{1+\omega_1}\left\{1 - \frac{i\delta(1+\omega_1)}{n\omega_1}\right\}\left[1 + \frac{\gamma\{4+\omega_0^2(1+\omega_1)^2\}}{(\gamma-1)\,\xi}\right] \\ & + \frac{4m^2}{n^2\omega_1\xi\{4+\omega_0^2(1+\omega_1)^2\}}\left\{1 - \frac{i\omega_0(1+\omega_1)}{m} + \frac{\omega_0^2(1+\omega_1)^2}{4}\right\} \\ & + \frac{\omega_0(1+\omega_1)}{n^2\omega_1\xi}\left[-\frac{2im(\gamma-2)}{\gamma-1} - 2\omega_0(1+\omega_1) + \frac{\omega_0(1+\omega_1)\{4+\omega_0^2(1+\omega_1)^2\}}{\gamma-1}\right. \\ & \left. + \frac{2\xi\{2im+4\omega_0(1+\omega_1)+\omega_0^3(1+\omega_1)^3\}}{(\xi+\gamma_1)\{4+\omega_0^2(1+\omega_1)^2\}} + \frac{\gamma m^2\omega_0(1+\omega_1)}{(\gamma-1)\,\xi}\right] \end{aligned} \tag{21}$$

and use has been made of the transformation

$$\xi = r^2, \quad \hat{p} = e^{-\frac{1}{4}\xi}q, \quad \omega = \omega_0(1+\omega_1), \quad \omega_0 = -\frac{m\delta}{2n}, \quad \alpha^2 = \frac{\gamma-1}{\gamma}\frac{n^2}{\omega_0^2}. \tag{22}$$

Motivated by the WKB method of approximation, let us apply the transformation

$$q = e^f. \tag{23}$$

The resulting equation is

$$4\left(\frac{df}{d\xi}\right)^2 - \left\{\frac{1}{4} + \frac{\omega_1\alpha^2A_0}{(1+\omega_1)^2} + A_1\right\} = -4\frac{d^2f}{d\xi^2} - 4A_1\frac{df}{d\xi}. \tag{24}$$

Let us assume here that ω_1 is of order ξ^{-1}, which will shortly be justified. Next, using the fact that δ is roughly $\frac{1}{3000}$ and ξ is about 100 for the centrifuge referred to in §1, let us assume that

$$\delta \ll 1, \quad \xi^{-1} \ll 1, \quad \xi\omega_0 \ll 1. \tag{25}$$

The above fact is the basis for our assumption that the axial component of the wavenumber vector is large in comparison with H^{-1} and that the radial pressure scale height is small in comparison with r. On the basis of (25), we can regard terms of order δ, ξ^{-1} and $\xi\omega_0$ as infinitesimal. For example, (18) can be simplified as follows:

$$\hat{u} = \frac{2\gamma\omega_0}{(\gamma-1)\,r}\left\{\frac{d\hat{p}}{d\xi} - \frac{im}{\xi\omega_0}\hat{p}\right\}. \tag{26}$$

Consistent with (25), α is large in comparison with unity. Therefore the right-hand side of (24) can be neglected in comparison with the left-hand side. Assumptions (25) are also used to simplify the second term on the left-hand side of (24). The solution of this approximated equation gives

$$\hat{p} = \exp\left(-\tfrac{1}{4}\xi\right)\left(A\exp f_0 + B\exp -f_0\right), \tag{27}$$

where
$$f_0 = \frac{\xi}{2}\left(\lambda^2+\frac{\beta^2}{\xi}\right)^{\frac{1}{2}} - \frac{\beta^2}{4\lambda}\log\left[\frac{(\lambda^2+\beta^2/\xi)^{\frac{1}{2}}-\lambda}{(\lambda^2+\beta^2/\xi)^{\frac{1}{2}}+\lambda}\right], \tag{28}$$
$$\lambda^2 = \tfrac{1}{4}+\alpha^2\omega_1, \quad \beta^2 = \alpha^2 m^2/n^2 \tag{29}$$
and A and B are non-vanishing coefficients.

Substitution of the solution (27) into the boundary conditions that u (the normal component of the velocity) vanishes on $r = r_1$ and r_2 gives
$$0 = g(\lambda_1) = (\lambda_1^2+1)^{\frac{1}{2}} - \frac{\xi_2}{\xi_1}\left(\lambda_1^2+\frac{\xi_1}{\xi_2}\right)^{\frac{1}{2}} - \log\left(\frac{\xi_2}{\xi_1}\right)^{\frac{1}{2}}\{(\lambda_1^2+1)^{\frac{1}{2}}-\lambda_1\}\left\{\left(\lambda_1^2+\frac{\xi_2}{\xi_1}\right)^{\frac{1}{2}}+\lambda_1\right\}, \tag{30}$$
where
$$\lambda_1 = \xi_1^{\frac{1}{2}}\lambda/\beta, \quad \xi_1 = r_1^2, \quad \xi_2 = r_2^2 \tag{31}$$
and we have neglected terms of small magnitude in accordance with (25). In the limiting case of a narrow gap, in which ξ_1/ξ_2 is expressed as
$$\xi_1/\xi_2 = 1-\epsilon_{12}, \quad \epsilon_{12} \ll 1, \tag{32}$$
we can show that the solution of (30) is of the form
$$\lambda_1 = -\tfrac{1}{4} \pm \tfrac{1}{4}\times 7^{\frac{1}{2}} i + O(\epsilon_{12}). \tag{33}$$
In view of this limiting form and the fact that the coefficients in (30) are all real, we can expect that the solutions of (30) are a pair of the complex conjugates one of which is in the second quadrant of the λ_1 plane. We can obtain the solution (in the second quadrant) from the following integral:
$$\lambda_1 = \frac{1}{2\pi i}\int_C \frac{\lambda g'(\lambda)}{g(\lambda)}\,d\lambda, \tag{34}$$
where the prime means differentiation with respect to λ and C is a contour encircling the expected location of the solution. The other solution can be obtained as the complex conjugate of (34). The integral is performed numerically along the contour shown in figure 2, and the results are summarized in table 1 and in figure 3.

By substituting these results into the full expression for ω,
$$\omega = -\frac{\delta m}{2n}\left(1+\frac{\lambda_1^2 m^2}{\xi_1 n^2}\right), \tag{35}$$
we see that our simplifying assumption that ω may be expressed by (22) and that ω_1 is of order ξ^{-1} is self consistent. For m/n positive (or for m/n negative and λ_1 replaced by its complex conjugate), the real part of the right-hand side is negative. By (6), this means that the mode is unstable. Because λ_1 is complex so is ω. This means that the unstable mode takes the form of a growing oscillation, that is to say, the mode is of overstable type. The growth time and the oscillation period of the mode are of orders δ^{-1} and $(\xi_1\delta)^{-1}$, respectively, that is to say of orders $(\delta\Omega)^{-1}$ and $(\xi_1\delta\Omega)^{-1}$, respectively, in the original dimensional expression.

Before concluding, it is to be noted that there exist at least two more cases of interest, which we purposely omitted for the sake of simplicity. These are that

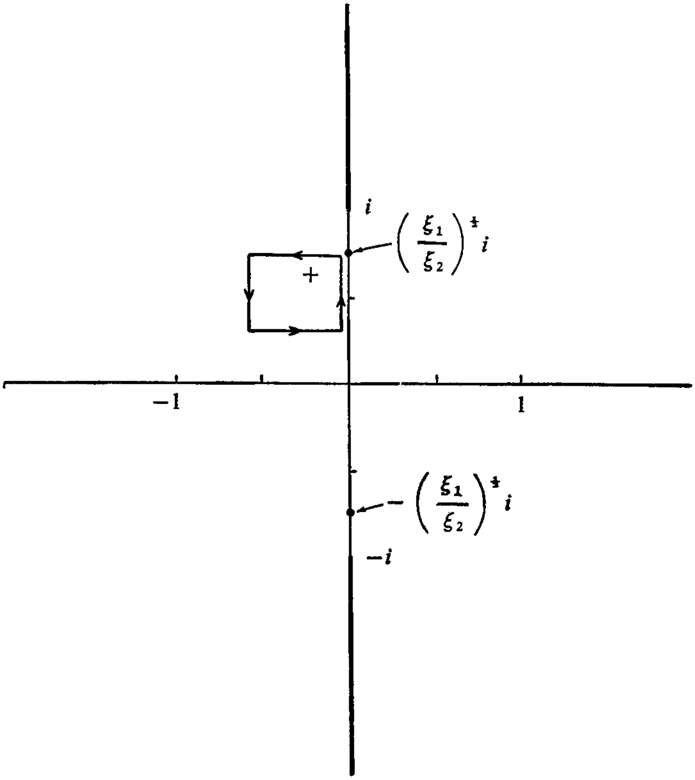

FIGURE 2. The contour in the λ_1 plane along which the integral (36) is evaluated. Because there exist branch points of $(\lambda_1^2+1)^{\frac{1}{2}}$ and of $(\lambda_1^2+\xi_1/\xi_2)^{\frac{1}{2}}$ at $\pm i$ and $\pm(\xi_1/\xi_2)^{\frac{1}{2}}\, i$, respectively, care must be taken in the calculation of these square roots. The plus sign on the contour is the expected location of the zero point.

ξ_1/ξ_2	$\mathscr{R}(\lambda_1)$	$\mathscr{I}(\lambda_1)$
1·00	−0·25	0·6614
0·95	−0·2417	0·6548
0·9	−0·2349	0·6453
0·85	−0·2282	0·6365
0·8	−0·2207	0·6277
0·75	−0·2127	0·6182
0·7	−0·2041	0·6079
0·65	−0·1950	0·5969
0·6	−0·1854	0·5851
0·55	−0·1757	0·5725
0·5	−0·1667	0·5578

TABLE 1

in which $\omega^2 \sim -4$, case 1, say, and that in which $\omega^2 \sim -\{4+r^2(\gamma-1)/\gamma\}$, case 2. Case 1 corresponds to resonance with a typical inertial oscillation in the rotating fluid (Greenspan 1968). Case 2 corresponds to resonance with a local Brunt–Väisälä oscillation in a radially stratified rotating fluid subject to a strong centrifugal force. As can be seen from (19), the simple method of treatment in

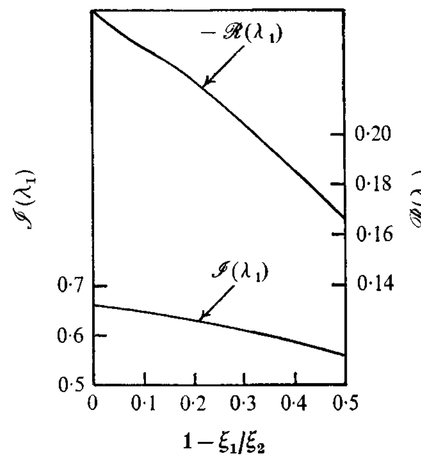

FIGURE 3. Real and imaginary parts of λ_1 as functions of ξ_1/ξ_2. Because λ_1 is a complex number, ω in (37) is also complex. That is to say, the mode is of overstable type.

this paper, i.e. a straightforward simplification based on (25) and the application of the WKB method of approximation (without a turning point), can not be applied to these cases. Because our aim in this paper is to draw attention to the baroclinic-type instability in a gas centrifuge heated from above (not from below), we omit the discussion of these complicated cases. The analysis of these cases will be given elsewhere.

This paper is dedicated to Professor Isao Imai in celebration of his sixtieth birthday. The author wishes to express his thanks to Professor Hidenori Hasimoto and Professor Isamu Hirota for valuable discussions. He is also indebted to Professor Sir James Lighthill and Dr Michael E. McIntyre for valuable suggestions for improving the clarity of the manuscript.

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